UNIT-05

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HYPER GEOMETRIC DISTRIBUTION

Used when sampling without replacement

Hyper Geometric Distribution:

A Discrete random variable X assumes the values 0, 1, 2, ... with the P.M.F

$$P(X=x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \; ; x = 0, 1, 2, \ldots \min(n, m)$$

N = Total no. of items in the population

M = no. of success items in the populations

is called hyper Geometric distribution with parameter (M, N, n)

we have

$$\sum_{x=0}^{n} \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} = 1$$

n = no. of items drawn

x = no. of succes obtained in the sample

Sampling from small populations without replacement

quality control

(or)

$$\sum_{x=0}^{n} \binom{M}{x}\binom{N-M}{n-x} = \binom{N}{n}$$

Mean and Variance of Hyper Geometric Distribution:

we know that

$$\mu_1' = E(X)$$

$$= \sum_{x=0}^{n} x \cdot p(x)$$

$$= \frac{\sum_{x=0}^{n} x \cdot \binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} = \frac{1}{\binom{N}{n}} \sum_{x=0}^{\infty} x\binom{M}{x}\binom{N-M}{n-x}$$

$$= \frac{1}{\binom{N}{n}} \sum_{x=0}^{n} \cancel{x} \frac{M}{x} \binom{M-1}{x-1}\binom{N-M}{n-x}$$

$$= \frac{M}{\binom{N}{n}} \sum_{x-1=0}^{n} \binom{M-1}{x-1}\binom{N-1-(M-1)}{n-1-(x-1)}$$

$$\mu_1' = \frac{M}{\binom{N}{n}} \binom{N-1}{n-1}$$

$$= \frac{M \binom{N-1}{n-1}}{\frac{N}{n}\binom{N-1}{n-1}} = \frac{Mn}{N}$$

$$\boxed{\text{Mean} = \frac{Mn}{N}} = \mu_1'$$

*Variance of Hyper Geometric Distribution:

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

consider, $E(X^2) = E[X(X-1) + X]$

$$= E[X(X-1)] + E(X)$$

$$= \sum_{x=0}^{n} x(x-1)\, p(x) + \frac{Mn}{N}$$

$$= \frac{\sum_{x=0}^{n} x(x-1) \binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} + \frac{Mn}{N}$$

$$E(x^2) = \frac{1}{\binom{N}{n}} \sum_{x=0}^{n} x(x-1) \frac{M(M-1)}{x(x-1)} \binom{M-2}{x-2}\binom{N-M}{n-x} + \frac{Mn}{N}$$

$$= \frac{M(M-1)}{\binom{N}{n}} \sum_{x-2=0}^{n} \binom{M-2}{x-2}\binom{N-2-(M-2)}{n-2-(x-2)} + \frac{Mn}{N}$$

$$E(x^2) = \frac{M(M-1)}{\binom{N}{n}} \binom{N-2}{n-2} + \frac{Mn}{N}$$

$$= \frac{M(M-1)\binom{N-2}{n-2}}{\frac{N(N-1)}{n(n-1)}\binom{N-2}{n-2}} + \frac{Mn}{N}$$

$$E(x^2) = \frac{M(M-1)\,n(n-1)}{N(N-1)} + \frac{Mn}{N}$$

$$Var(x) = E(x^2) - [E(x)]^2 = \frac{M(M-1)n(n-1)}{N(N-1)} + \frac{Mn}{N} - \left(\frac{Mn}{N}\right)^2$$

$$= \frac{Mn}{N}\left[\frac{(M-1)(n-1)}{(N-1)} + 1\right] - \left(\frac{Mn}{N}\right)^2$$

$$= \frac{Mn}{N}\left[\frac{(M-1)(n-1)}{(N-1)} + 1 - \frac{Mn}{N}\right]$$

$$= \frac{Mn}{N}\left[\frac{N(Mn-M-n+1) + N(N-1) - Mn(N-1)}{N(N-1)}\right]$$

$$= \frac{Mn}{N^2(N-1)}\left[NMn - NM - Nn + N + N^2 - N - MnN + Mn\right]$$

$$= \frac{Mn}{N^2(N-1)}\left[N^2 - NM - Nn + Mn\right]$$

$$= \frac{Mn}{N^2(N-1)} [N(N-M) - n(N-M)]$$

$$= \frac{Mn}{N^2(N-1)} (N-n)(N-M)$$

$$\boxed{Var(x) = \frac{Mn(N-n)(N-M)}{N^2(N-1)}}$$

* Recurrence relation for probability of Hyper Geometric distribution:

$$p(x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

$$p(x+1) = \frac{\binom{M}{x+1}\binom{N-M}{n-x-1}}{\binom{N}{n}}$$

$$\frac{p(x+1)}{p(x)} = \frac{\binom{M}{x+1}\binom{N-M}{N-x-1}}{\binom{N}{n}} \cdot \frac{\binom{N}{n}}{\binom{M}{x}\binom{N-M}{n-x}}$$

$$= \frac{M!}{(x+1)!(M-x-1)!} \cdot \frac{x!(M-x)!}{M!} * \frac{N-M}{(n-x-1)!(N-M-n+x+1)}$$

$$= \frac{(n-x)(M-x)}{(x+1)(N-M-n+x+1)}$$

$$p(x+1) = \frac{(n-x)(M-x)}{(x+1)(N-M-n+x+1)} \cdot p(x)$$

* Binomial distribution as a limiting case of hyper geometric distribution:

Hyper Geometric distribution tends to Binomial distribution under the following conditions.

1. $N \to \infty$

2. $\frac{M}{N} \to \infty$ and $1 - \frac{M}{N} = q$

$$\lim_{N\to\infty} p(x) = \lim_{N\to\infty} \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

$$= \lim_{N\to\infty} \frac{M!}{x!(M-x)!} \cdot \frac{(N-M)!}{(N-M-n+x)!\,(n-x)!} \cdot \frac{n!\,(N-n)!}{N!}$$

$$= \lim_{N\to\infty} \frac{M(M-1)\cdots(M-x+1)}{x!} \cdot \frac{(N-M)(N-M-1)\cdots(N-M-n+x+1)}{(n-x)!} \cdot \frac{n!}{N(N-1)\cdots(N-n+1)}$$

$$= \lim_{N\to\infty} \frac{\frac{M}{N}\left(\frac{M}{N}-\frac{1}{N}\right)\cdots\left(\frac{M}{N}-\frac{x-1}{N}\right) \times \left(1-\frac{M}{N}\right)\left(1-\frac{M}{N}-\frac{1}{N}\right)\cdots\left(1-\frac{M}{N}-\frac{n-x-1}{N}\right)}{\frac{N}{N}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{n-1}{N}\right)}$$

as $N \to \infty$, $\frac{1}{N} \to \infty$ and $\frac{M}{N} \to p$

$$\binom{n}{x} \underbrace{(p \cdot p \cdots p)}_{x \text{ terms}} \underbrace{(1-p)(1-p) \cdots (1-p)}_{(n-x) \text{ terms}}$$

$$\Rightarrow \binom{n}{x} p^x q^{n-x}$$

This is p.M.F of Binomial distribution.

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